# Homework I

## Small Benders decomposition example with feasibility cuts

Consider the MIP

|  |  |
| --- | --- |
|  | ( ) |
|  | ( ) |
|  | ( ) |
|  | ( ) |
|  | ( ) |

* 1. What is the master problem of a Benders decomposition applied to this problem, and what is the Benders subproblem in its dual form (here ) ?

Graphic representation of the problem is set on the following figure:

Chart

Description automatically generated

Figure : Graphic Representation of the Optimization Problem. The feasible region is depicted in brown, and its three vertices denoted A, B and C with their coordinates.

It is immediately apparent that the only admissible region for the linear constraints is the small triangle in the middle of the three regions and that the only feasible solution is – since it is the only point in the feasible region where .

We have a problem of the form:

|  |  |
| --- | --- |
|  | ( ) |
|  | ( ) |

Where we have:

* , , , et .

According to the slide 4 of course 4, the Benders decomposition gives us

And if we replace the subproblem by its dual, we have:

Where .

The Benders master problem is

|  |  |
| --- | --- |
|  | ( ) |

In this case, it is apparent from Figure 1 that the objective function value lies in the interval . Since , a first lower bound for could be .

|  |  |
| --- | --- |
|  | ( ) |

The Benders subproblem in its dual form is:

|  |  |
| --- | --- |
|  | ( ) |

Which translates into:

|  |  |
| --- | --- |
|  | ( ) |

* 1. Show that for the candidate master solution , the Benders subproblem (dual form) is unbounded and give a corresponding extreme ray. Write the corresponding feasibility cut. Give an interpretation of this constraint on in view of the constraints ( 2 ) and ( 4 ). Are there other feasibility cuts?

If we take the master problem, we have:

|  |  |
| --- | --- |
|  | ( ) |
|  | ( ) |

And we have that the solution is minimal whenever and equal to . The next step is to find a lower bound for by solving the following subproblem:

|  |  |
| --- | --- |
|  | ( ) |

We identify the planes related to the constraints and the vertexes as depicted on the following figure:

Chart, radar chart

Description automatically generated

Figure : Representation of Dual Subproblem. The feasible set is represented by the convex hull containing (in orange) and the cone of the extreme rays (in blue).

We first denote the vertices:

And then the extreme rays ():

We observe that the admissible set for this maximization problem is the union of the convex hull of vertices and the cone of extreme rays .

The problem is unbounded, because if we start from anywhere in the convex hull of and follow direction , the constraints are all respected and the value of the objective function can be arbitrarily increased. Taking for example as start point and imposing the research direction with , one has and the problem becomes

|  |  |
| --- | --- |
|  | ( ) |

And we see that the constraints are indeed respected, and the objective function is unbounded. Note that this is the only direction composing the cone in which the objective function is unbounded, and the constraints remain satisfied.

In this case, it is necessary to add a “feasibility cut” to the primal problem by adding a constraint:

Which translated into:

It is interesting to remark that the vector indicates the constraints which bounds the value of , namely ( 2 ) and ( 4 ), and actually points out the value as inacceptable. In practice, feasibility cuts ensures that minimization and computation of a value for in the master problem of the Benders decomposition leaves a feasible value of in the subproblem.

* 1. Add to the master problem all possible optimality cuts. Hint: Analyse the feasible set of the (dual) Benders subproblem and identify its vertices.

The optimality cut are defined by the vertexes we identified in the convex set in the previous point namely and we have as an optimality cut:

We then have

And applying the value , we have

* 1. Draw the optimality cuts (in the space ) and give an interpretation.

On the following figure, we represented the feasible region in the plane.

Chart, line chart

Description automatically generated

Figure : Feasible region for the Benders Decomposition of the Problem (in brown), feasibility cut (in blue) and optimality cuts (in red).

The objective function is minimized when one is following the direction with . It appears that the function is minimized when we have if the integer constraint is relaxed. Otherwise, the best result is given by as we expected from the first point.

For the Benders dual subproblem, the feasibility cut added to the master problem in ( 9 ) allows to make sure that the subproblem problem in ( 11 ) remains bounded. The idea is to impose a condition on that makes sure that the objective function of the subproblem cannot be increased arbitrarily in any feasible direction (the extreme rays ). If we call the set of feasible points of the original problem, the feasibility cut describes a set in the Benders decomposition such that .

The optimality cuts, on the other hand, are used to define lower bounds on the dual subproblem optimal solution. If this problem is bounded, the vector that maximizes the Benders dual subproblem objective function, is a vertex of the polytope describing the feasible region: . So by minimizing the objective function while keeping it above the value encountered everywhere in , we find the optimal value of the subproblem.

Once the Benders dual subproblem is bounded (by adding feasibility cuts), the optimal solution is one of the vertexes or a linear combination of them. By adding all optimality cuts (taking equal to the maximum value of the objective function evaluated at all vertexes) one computes the optimal value of the Benders dual subproblem.

## Solving the farmer’s problem example via the L-shaped method

Consider the stochastic optimization program of the farmer’s problem given on slide 12 of the slide deck “Course 5 - Introduction to Stochastic Programming via examples”.

* 1. Solve the problem via the L-shaped method.
  2. Report on the optimality and feasibility cuts generated when solving the problem. Are there feasibility cuts? Can we have anticipated this in view of the structure of the initial problem (problem in extensive form) ?

We expect no feasibility cut in this problem because the slave problem is always feasible. Indeed, no matter the first-stage decision, second-stage decision have enough degrees of freedom to satisfy the constraint. For the two first constraints (where the independent term is positive) we can easily adapt the value on the right hand-side term with the variables and .

In the problem terms, it means that no matter what has been produced, if resources are lacking, it is always possible to buy additional resources to satisfy the requirements (independently from what has been planted – resources 1 and 2/wheat and corn). For the very resources it is impossible to buy, there are no requirements (resource 3/sugar beets)

## Solving a capacity expansion planning problem via the L-shaped method

Consider the capacity expansion planning problem described in Section 1.2 of the course textbook (Anthony Papavasiliou, Stochastic Dual Dynamic Programming, manuscript, see the file “text-book.pdf” on Moodle) and on slide 9 of the course on Performance of Stochastic Programming Solutions.

* 1. Implement the extensive form of the corresponding stochastic optimization problem (see the textbook page 10 for the general formulation), solve it directly and verify that you obtain the same solution as in the notes.
  2. Now, re-solve the extensive form program assuming that you cannot use demand response, i.e. the last technology ”DR” is not available, and report the results.
  3. Do we expect to have to generate feasibility cuts if we solve this new problem (without allowing for demand response) via the L-shaped method ? Why ?
  4. Solve the problem via the L-shaped method and report on the feasibility and optimality cuts generated during the resolution.